$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l}\, dx.$$ Applying Bernoulli's rule,

$$= \frac{2}{l} \left[(lx - x^2) \cdot \frac{-\cos \frac{n\pi x}{l}}{(n\pi/l)} - (l - 2x) \cdot \frac{-\sin \frac{n\pi x}{l}}{(n\pi/l)^2} + (-2) \frac{\cos \frac{n\pi x}{l}}{(n\pi/l)^3} \right]_0^l$$

$$= \frac{-4}{l} \cdot \frac{l^3}{n^3 \pi^3} \left[\cos \frac{n\pi x}{l} \right]_0^l$$

($lx - x^2$ *is zero at* $x = 0,\ l$ *and* $\sin n\pi = 0 = \sin 0$)

$$b_n = \frac{-4\, l^2}{n^3 \pi^3} (\cos n\pi - 1) = \frac{4\, l^2}{n^3 \pi^3} \{1 - (-1)^n\}$$

The sine half range Fourier series is given by

$$f(x) = \frac{4\, l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \{1 - (-1)^n\} \sin \frac{n\pi x}{l}$$

But $$1 - (-1)^n = \begin{cases} 1 - (+1) = 0 & \text{when } n \text{ is even} \\ 1 - (-1) = 2 & \text{when } n \text{ is odd} \end{cases}$$

$\therefore$ $$f(x) = \frac{4\, l^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{2}{n^3} \sin \frac{n\pi x}{l}$$

ie., $$f(x) = \frac{8 l^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{l}$$

But 1, 3, 5, . . . are odd numbers represented in general as $(2n + 1)$ where $n = 0, 1, 2, 3, \ldots$. Thus we have,

$$f(x) = \frac{8\, l^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\left(\frac{2n+1}{l} \right) \pi x$$

34. *Show that* $\frac{l}{2} - x = \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi x}{l}\right), \quad 0 < x < l$

>> From the given answer and the given interval of x it is evident that we have to find the sine half range Fourier series of $f(x) = \frac{l}{2} - x$ in $(0, l)$.

The series is represented by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

i.e., $$b_n = \frac{2}{l}\int_0^l \left(\frac{l}{2} - x\right)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{l}\left[\left(\frac{l}{2} - x\right)\cdot\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)} - (-1)\cdot\frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^2}\right]_0^l$$

$$= \frac{-2}{l}\,\frac{l}{n\pi}\left[\left(\frac{l}{2} - x\right)\cos\frac{n\pi x}{l}\right]_0^l, \quad \text{since } \sin n\pi = 0 = \sin 0$$

$$= \frac{-2}{n\pi}\left(-\frac{l}{2}\cos n\pi - \frac{l}{2}\right) = \frac{l}{n\pi}(\cos n\pi + 1)$$

$$\boldsymbol{b_n = \frac{l}{n\pi}\left[1 + (-1)^n\right]}$$

The sine half range Fourier series is given by

$$f(x) = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + (-1)^n\right]\sin\frac{n\pi x}{l}$$

But $$1 + (-1)^n = \begin{cases} 1 + 1 = 2 \text{ when } n \text{ is even} \\ 1 - 1 = 0 \text{ when } n \text{ is odd} \end{cases}$$

$\therefore$ $$f(x) = \frac{l}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{2}{n}\sin\frac{n\pi x}{l}$$

But 2, 4, 6,.... are even numbers represented in general as $2n$ where $n = 1, 2, 3, \ldots$ and hence we can write the series replacing n by $2n$ in the form,

$$\boldsymbol{f(x) = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}}$$

---

35. *Find the cosine half range series for* $f(x) = x(l-x);\ 0 \le x \le l$

>> The cosine half range Fourier series of $f(x)$ in $[0,\ l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \text{ where}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx\,;\quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{2}{l}\int_0^l (lx - x^2)\,dx = \frac{2}{l}\left[l\,\frac{x^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left(\frac{l^3}{2} - \frac{l^3}{3}\right) = \frac{l^2}{3}$$

$$a_0/2 = l^2/6$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l}\left[(lx - x^2)\frac{\sin\frac{n\pi x}{l}}{(n\pi/l)} - (l-2x)\cdot\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)^2} + (-2)\cdot\frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^3}\right]_0^l$$

$$= \frac{2}{l}\,\frac{l^2}{n^2\pi^2}\left[(l-2x)\cos\frac{n\pi x}{l}\right]_0^l \quad \text{since first and third terms vanish.}$$

$$= \frac{2l}{n^2\pi^2}(-l\cos n\pi - l) = \frac{-2l^2}{n^2\pi^2}(\cos n\pi + 1)$$

$$a_n = \frac{-2l^2}{n^2\pi^2}\{1 + (-1)^n\}$$

Thus the required cosine half range series is given by

$$f(x) = \frac{l^2}{6} - \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\{1 + (-1)^n\}\cos\frac{n\pi x}{l}$$

---

36. *Obtain the half range cosine series for the function*

$$f(x) = \sin\left(\frac{m\pi x}{l}\right) \textit{ where } m \textit{ is a positive integer over the interval } (0, l)$$

>> We have $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l}$ where

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\,dx$$

$$= \frac{2}{l}\cdot\frac{l}{m\pi}\left[-\cos\frac{m\pi x}{l}\right]_0^l = \frac{-2}{m\pi}(\cos m\pi - 1) = \frac{-2}{m\pi}\left[(-1)^m - 1\right]$$

$$\frac{a_0}{2} = \frac{1}{m\pi}\left\{1 - (-1)^m\right\}$$

$$a_n = \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\cdot\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \frac{1}{2}\left\{\sin(m+n)\frac{\pi x}{l} + \sin(m-n)\frac{\pi x}{l}\right\}dx$$

$$= \frac{1}{l}\left[\frac{-l}{(m+n)\pi}\cos(m+n)\frac{\pi x}{l} - \frac{l}{(m-n)\pi}\cos(m-n)\frac{\pi x}{l}\right]_0^l, \quad m \neq n$$

$$= \frac{1}{\pi}\left[\frac{-1}{(m+n)}\left\{\cos(m+n)\pi - 1\right\} - \frac{1}{m-n}\left\{\cos(m-n)\pi - 1\right\}\right]$$

$$= \frac{-1}{\pi}\left[\left\{\frac{-1}{m+n} - \frac{1}{m-n}\right\} + \frac{1}{m+n}\left(\cos m\pi\cos n\pi - \sin m\pi\sin n\pi\right) + \frac{1}{(m-n)}\left(\cos m\pi\cos n\pi + \sin m\pi\sin n\pi\right)\right]$$

$$= \frac{-1}{\pi}\left[-\left\{\frac{1}{m+n} + \frac{1}{m-n}\right\} + \cos m\pi\cos n\pi\left\{\frac{1}{m+n} + \frac{1}{m-n}\right\}\right]$$

where we have $\sin m\pi = 0 = \sin n\pi$

$$\therefore \quad a_n = \frac{1}{\pi}\left[\frac{2m}{m^2-n^2}\{1-\cos m\pi \cos n\pi\}\right]$$

$$a_n = \frac{2m}{\pi(m^2-n^2)}\left\{1+(-1)^{m+n+1}\right\} \text{ where } m \neq n$$

If $\quad m = n,\ a_n = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\cos\frac{n\pi x}{l}\,dx$

i.e., $\quad a_n = \frac{1}{l}\int_0^l \sin\frac{2n\pi x}{l}\,dx = \frac{1}{l}\left[\frac{-l}{2n\pi}\cos\frac{2n\pi x}{l}\right]_0^l$

$$= -\frac{1}{2n\pi}(\cos 2n\pi - 1) = 0, \text{ since } \cos 2n\pi = +1$$

$a_n = 0$ **when** $m = n$.

Thus the required cosine half range Fourier series when $m \neq n$ is given by

$$f(x) = \frac{1}{m\pi}\left\{1-(-1)^m\right\} + \sum_{n=1}^{\infty}\frac{2m}{\pi(m^2-n^2)}\left\{1+(-1)^{m+n+1}\right\}\cos\frac{n\pi x}{l}$$

---

37. *Find a cosine series for* $f(x) = (x-1)^2,\ 0 \le x \le 1$

>> Comparing the given interval [0, 1] with half range $[0,\ l]$ we have $l = 1$. The corresponding cosine half range Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x, \text{ where}$$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx;\ a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\,dx$$

$$a_0 = 2\int_0^1 (x-1)^2\,dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3}\{0-(-1)^3\} = \frac{2}{3}$$

$a_0/2 = 1/3$

$$a_n = 2\int_0^1 (x-1)^2 \cos n\pi x \, dx.$$

$$= 2\left[(x-1)^2 \cdot \frac{\sin n\pi x}{n\pi} - 2(x-1)\cdot -\frac{\cos n\pi x}{n^2\pi^2} + 2 \cdot -\frac{\sin n\pi x}{n^3\pi^3} \right]_0^1$$

$$= \frac{4}{n^2\pi^2}\Big[(x-1)\cos n\pi x \Big]_0^1 = \frac{4}{n^2\pi^2}\{0-(-1)\} = \frac{4}{n^2\pi^2}$$

$$\boldsymbol{a_n = 4/n^2\pi^2}$$

Thus the required cosine half range Fourier series is

$$\boldsymbol{f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\cos n\pi x}$$

Note : *By putting* $x = 0$ *and* $x = 1$ *in the series we can respectively deduce the series*

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots ; \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Adding these we also obtain $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

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38. *Find the cosine half range series of* $f(x) = x\sin x$ *in* $0 < x < \pi$. *Deduce that*

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots = \frac{\pi-2}{4} \quad or \quad 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \cdots = \frac{\pi}{2}$$

>> The cosine half range series of $f(x)$ in half the range $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where}$$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

i.e., $$a_0 = \frac{2}{\pi}\int_0^\pi x\sin x\,dx \; ; \quad a_n = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx$$

Referring to Problem-14 for the integration process, the required series is given by

$$\boldsymbol{f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx}$$

Now putting $x = \pi/2$, $f(x) = \pi/2 \sin(\pi/2) = \pi/2$

The Fourier series becomes

$$\frac{\pi}{2} = 1 + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cos\frac{n\pi}{2}$$

$$\frac{\pi}{2} - 1 = 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cos\frac{n\pi}{2} \quad \text{or} \quad \frac{\pi-2}{2} = 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cos\frac{n\pi}{2}$$

On expanding the R.H.S and using $\cos\pi = -1 = \cos 3\pi = \cos 5\pi \ldots$

$\cos(3\pi/2) = 0 = \cos(5\pi/2) \cdots$ we get

$$\frac{\pi-2}{4} = \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \cdots,$$

Thus $$\boldsymbol{\frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots}$$

or

Multiplying by 2 and transposing 1 on to the R.H.S we get

$$\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots\cdots$$

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39. *If* $f(x) = \begin{cases} x & \text{in } 0 < x < \dfrac{\pi}{2} \\ \pi - x & \text{in } \dfrac{\pi}{2} < x < \pi \end{cases}$ *show that*

(i) $f(x) = \dfrac{4}{\pi}\left[\sin x - \dfrac{\sin 3x}{3^2} + \dfrac{\sin 5x}{5^2} - \cdots\right]$

(ii) $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\left[\dfrac{\cos 2x}{1^2} + \dfrac{\cos 6x}{3^2} + \dfrac{\cos 10x}{5^2} + \cdots\right]$

>> $f(x)$ is defined in $(0, \pi)$ and we need to find the Fourier sine half range series and cosine half range series separately.

(i) The sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

i.e., $$b_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} x\sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right\}$$

Applying Bernoulli's rule to each of the integrals we have

$$b_n = \frac{2}{\pi}\left\{\left[x \cdot -\frac{\cos nx}{n} - (1)\cdot -\frac{\sin nx}{n^2}\right]_0^{\pi/2} + \left[(\pi - x)\cdot \frac{-\cos nx}{n} - (-1)\cdot \frac{-\sin nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-1}{n}[x\cos nx]_0^{\pi/2} + \frac{1}{n^2}[\sin nx]_0^{\pi/2} - \frac{1}{n}\left[(\pi - x)\cos nx\right]_{\pi/2}^{\pi} - \frac{1}{n^2}[\sin nx]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-1}{n}\left(\frac{\pi}{2}\cos\frac{n\pi}{2} - 0\right) + \frac{1}{n^2}\left(\sin\frac{n\pi}{2} - 0\right) - \frac{1}{n}\left(0 - \frac{\pi}{2}\cos\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(0 - \sin\frac{n\pi}{2}\right)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{2}{\pi}\cdot\frac{2}{n^2}\sin\frac{n\pi}{2}$$

$$\mathbf{b_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}}$$

The required sine half range series is given by

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx$$

$$f(x) = \frac{4}{\pi}\left\{\frac{1}{1^2}\sin\frac{\pi}{2}\sin x + \frac{1}{2^2}\sin\pi\sin 2x + \frac{1}{3^2}\sin\frac{3\pi}{2}\sin 3x + \cdots\right\}$$

$$\mathbf{f(x) = \frac{4}{\pi}\left\{\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right\}}$$

(ii) The cosine half range Fourier series of $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx, \text{ where}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \;; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\left\{\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$a_0 = \frac{2}{\pi}\left\{\left(\frac{\pi^2}{8} - 0\right) + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right\} = \frac{\pi}{2}$$

$$\mathbf{a_0 / 2 = \pi / 4}$$

$$a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right\}$$

Applying Bernoulli's rule to each of the integrals,

$$a_n = \frac{2}{\pi}\left\{\left[x \cdot \frac{\sin nx}{n} - (1) \cdot -\frac{\cos nx}{n^2}\right]_0^{\pi/2} + \left[(\pi - x)\frac{\sin nx}{n} - (-1) \cdot -\frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left[x\sin nx\right]_0^{\pi/2} + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi/2} + \frac{1}{n}\left[(\pi - x)\sin nx\right]_{\pi/2}^{\pi} - \frac{1}{n^2}\left[\cos nx\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left(\frac{\pi}{2}\sin\frac{n\pi}{2} - 0\right) + \frac{1}{n^2}\left(\cos\frac{n\pi}{2} - 1\right) + \frac{1}{n}\left(0 - \frac{\pi}{2}\sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\left(\cos\frac{n\pi}{2} - 1 - \cos n\pi + \cos\frac{n\pi}{2}\right) - \frac{\pi}{2n}\sin\frac{n\pi}{2}\right\}$$

$$= \frac{2}{\pi n^2}\left(-1-\cos n\pi + 2\cos\frac{n\pi}{2}\right) = \frac{-2}{\pi n^2}\left\{1+(-1)^n - 2\cos\frac{n\pi}{2}\right\}$$

But $\quad 1+(-1)^n = 2$ when n is even

$\qquad\qquad = 0$ when n is odd

$$a_n = \frac{-2}{\pi n^2}\left(2-2\cos\frac{n\pi}{2}\right) \text{ where } n = 2, 4, 6, \ldots$$

$$a_n = \frac{-4}{\pi n^2}\left(1-\cos\frac{n\pi}{2}\right), \; n = 2, 4, 6, \ldots$$

But $\quad 1-\cos\frac{n\pi}{2} = \begin{cases} 1-(-1) = 2 \text{ where } n = 2, 6, 10, \ldots \\ 1-(+1) = 0 \text{ where } n = 4, 8, 12, \ldots \end{cases}$

$\therefore \quad a_n = \frac{-4}{\pi n^2}(2) = \frac{-8}{\pi n^2}$ where $n = 2, 6, 10, \ldots$

The required cosine half range Fourier series is given by

$$f(x) = \frac{\pi}{4} - \frac{8}{\pi}\sum_{n=2,6,10,\ldots}^{\infty}\frac{1}{n^2}\cos nx$$

ie., $\quad f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left(\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \cdots\right)$

$$= \frac{\pi}{4} - \frac{8}{\pi}\cdot\frac{1}{2^2}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \cdots\right)$$

$\therefore \quad \mathbf{f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right)}$

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40. *Obtain the sine half range series of*

$$f(x) = \begin{cases} \frac{1}{4} - x \; in \; 0 < x < \frac{1}{2} \\ x - \frac{3}{4} \; in \; \frac{1}{2} < x < 1 \end{cases}$$

>> $f(x)$ is defined in $(0, 1)$. Comparing with half the range $(0, l)$ we have $l = 1$. The corresponding sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{where } b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x \, dx$$

i.e., $$b_n = 2\left\{\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin n\pi x\,dx + \int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin n\pi x\,dx\right\}$$

Applying Bernoulli's rule to each of the integrals,

$$b_n = 2\left\{\left[\left(\frac{1}{4}-x\right)\cdot\frac{-\cos n\pi x}{n\pi} - (-1)\cdot\frac{-\sin n\pi x}{n^2\pi^2}\right]_0^{1/2} + \left[\left(x-\frac{3}{4}\right)\cdot\frac{-\cos n\pi x}{n\pi} - 1\cdot\frac{-\sin n\pi x}{n^2\pi^2}\right]_{1/2}^{1}\right\}$$

$$= 2\left\{\frac{-1}{n\pi}\left[\left(\frac{1}{4}-x\right)\cos n\pi x\right]_0^{1/2} - \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^{1/2} - \frac{1}{n\pi}\left[\left(x-\frac{3}{4}\right)\cos n\pi x\right]_{1/2}^{1} + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_{1/2}^{1}\right\}$$

$$= 2\left\{\frac{-1}{n\pi}\left(-\frac{1}{4}\cos\frac{n\pi}{2} - \frac{1}{4}\right) - \frac{1}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right) - \frac{1}{n\pi}\left(\frac{1}{4}\cos n\pi + \frac{1}{4}\cos\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\left(0 - \sin\frac{n\pi}{2}\right)\right\}$$

$$= 2\left\{\frac{1}{4n\pi}\left(\cos\frac{n\pi}{2} + 1 - \cos n\pi - \cos\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= 2\left\{\frac{1}{4n\pi}(1-\cos n\pi) - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$\mathbf{b_n = \frac{1}{2n\pi}\left\{1-(-1)^n\right\} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

Thus the sine half range series is given by

$$\mathbf{f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\left\{1-(-1)^n\right\} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin n\pi x}$$

---

41. *Obtain the sine half range Fourier series for the function*

$$f(x) = \begin{cases} \dfrac{2kx}{l} & in \;\; 0 \le x \le \dfrac{l}{2} \\ \dfrac{2k}{l}(l-x) & in \;\; \dfrac{l}{2} \le x \le l \end{cases}$$

>> $f(x)$ is defined in $(0,\ l)$ and the sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{2}{l}\left\{\int_0^{l/2}\frac{2kx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{4k}{l^2}\left\{\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

On integration [*similar to Problem - 39 (i)*] we obtain $b_n = \dfrac{8k}{n^2\pi^2}\sin\dfrac{n\pi}{2}$

Thus the required sine half range series is given by

$$\boldsymbol{f(x) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}}$$

---

42. *Obtain the half range cosine series for the function*

$$f(x) = \begin{cases} \dfrac{\pi}{3}, & 0 < x < \dfrac{\pi}{3} \\ 0, & \dfrac{\pi}{3} < x < \dfrac{2\pi}{3} \\ \dfrac{-\pi}{3}, & \dfrac{2\pi}{3} < x < \pi \end{cases}$$

>> $f(x)$ is defined in $(0,\ \pi)$ and the cosine half range series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{where}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \;; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\left\{\int_0^{\pi/3} \frac{\pi}{3}\,dx + \int_{\pi/3}^{2\pi/3} 0\,dx + \int_{2\pi/3}^{\pi} \frac{-\pi}{3}\,dx\right\}$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{3}\left\{\int_0^{\pi/3} 1\,dx - \int_{2\pi/3}^{\pi} 1\,dx\right\}$$

$$= \frac{2}{3}\left\{[\,x\,]_0^{\pi/3} - [\,x\,]_{2\pi/3}^{\pi}\right\}$$

$$= \frac{2}{3}\left\{\left(\frac{\pi}{3} - 0\right) - \left(\pi - \frac{2\pi}{3}\right)\right\} = 0$$

$$a_0 / 2 = 0$$

$$a_n = \frac{2}{\pi}\cdot\frac{\pi}{3}\left\{\int_0^{\pi/3} \cos\, n\,x\,dx - \int_{2\pi/3}^{\pi} \cos nx\,dx\right\}$$

$$= \frac{2}{3}\left\{\left[\frac{\sin nx}{n}\right]_0^{\pi/3} - \left[\frac{\sin nx}{n}\right]_{2\pi/3}^{\pi}\right\}$$

$$= \frac{2}{3n}\left\{\left(\sin\frac{n\pi}{3} - 0\right) - \left(0 - \sin\frac{2n\pi}{3}\right)\right\}$$

$$a_n = \frac{2}{3n}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)$$

Thus the cosine half range series is given by

$$f(x) = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)\cos nx$$

EXERCISES

Obtain the cosine half range Fourier series for the following functions over the indicated interval.

1. $f(x) = \left(1 - \frac{x}{l}\right)^2$ in $(0,\ l)$

2. $f(x) = 2x - 1$ in $(0,\ 2)$

3. $f(x) = \cos ax$, a is not an integer in $(0,\ \pi)$

12

4. $f(x) = \begin{cases} \frac{1}{4} - x & \text{in } 0 < x < \frac{1}{2} \\ x - \frac{3}{2} & \text{in } \frac{1}{2} < x < 1 \end{cases}$ 5. $f(x) = \begin{cases} x & \text{in } 0 < x < 4 \\ 8 - x & \text{in } 4 < x < 8 \end{cases}$

Obtain the sine half range series for the following functions over the indicated interval.

6. $f(x) = 2x - 1$ in $0 < x < 1$ 7. $f(x) = x^2 - x$ in $0 < x < 1$

8. $f(x) = 1 - \left(\frac{x}{\pi}\right)$ in $0 < x < \pi$ 9. $f(x) = \begin{cases} x & \text{in } 0 \le x \le \frac{1}{2} \\ 1 - x & \text{in } \frac{1}{2} \le x < 1 \end{cases}$

10. $f(x) = \begin{cases} kx & \text{in } 0 \le x \le \frac{l}{2}. \\ k(l - x) & \text{in } \frac{l}{2} \le x < l \end{cases}$

11. Show that the half range Fourier sine series for $f(x) = k$ in $(0, \pi)$ is $\frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$ and hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

12. Show that the cosine half range series for $f(x) = a \sin x$ in $(0, \pi)$ is $\frac{4a}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{1 \cdot 3} - \frac{\cos 4x}{3 \cdot 5} - \frac{\cos 6x}{5 \cdot 7} \cdots\right)$ and hence deduce that $\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \cdots = \frac{\pi - 2}{4}$

ANSWERS

1. $\frac{1}{3} + \frac{4}{\pi^2} \sum_1^\infty \frac{1}{n^2} \cos \frac{n\pi x}{l}$

2. $1 - \frac{8}{\pi^2} \sum_1^\infty \left\{1 - (-1)^n\right\} \cos \frac{n\pi x}{2}$

3. $\frac{\sin a\pi}{a\pi} + \frac{2a}{\pi} \sum_1^\infty (-1)^{n+1} \frac{\sin a\pi \cos nx}{n^2 - a^2}$

4. $\frac{2}{\pi^2} \sum_1^\infty \frac{1}{n^2} \left\{1 + \cos n\pi - 2\cos \frac{n\pi}{2}\right\} \cos n\pi x$

5. $\frac{-16}{\pi^2} \sum_1^\infty \frac{1}{n^2} \left\{ 1 + \cos n\pi - 2\cos \frac{n\pi x}{8} \right\}$

6. $\frac{-4}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{1}{n} \sin n\pi x$

7. $\frac{-8}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} \sin 2n\pi x$

8. $\frac{2}{\pi} \sum_1^\infty \frac{1}{n} \sin nx$

9. $\frac{4}{\pi^2} \sum_1^\infty \frac{1}{n^2} \sin \frac{n\pi}{2} \sin n\pi x$

10. $\frac{4kl}{\pi^2} \sum_1^\infty \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l}$

1.9 Complex form of Fourier series

If $f(x)$ is a periodic function of period 2π defined in $(c, c+2\pi)$ we have the Fourier series of $f(x)$ given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

where a_0, a_n, b_n are given by Euler's formulae.

We know that,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{-i}{2}\left(e^{i\theta} - e^{-i\theta}\right)$$

Hence (1) becomes,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{inx} + e^{-inx}\right) + \sum_{n=1}^{\infty} \frac{-b_n i}{2}\left(e^{inx} - e^{-inx}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right) e^{inx} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2}\right) e^{-inx}$$

Let $C_0 = \frac{a_0}{2}, \; C_n = \frac{a_n - ib_n}{2}, \; C_n' = \frac{a_n + ib_n}{2}$ ... (2)

$$\therefore \quad f(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{inx} + \sum_{n=1}^{\infty} C_n' e^{-inx} \qquad \ldots (3)$$

We have the following basic integral,

$$\int_c^{c+2\pi} e^{\pm ikx}\,dx = \frac{1}{\pm ik}\left[e^{\pm ikx}\right]_c^{c+2\pi} = \frac{1}{\pm ik}\left[\cos kx \pm i\sin kx\right]_c^{c+2\pi} = 0$$

Integrating (3) w.r.t x between $c,\ c+2\pi$ we have

$$\int_c^{c+2\pi} f(x)\,dx = C_0\int_c^{c+2\pi} dx + 0 + 0 = C_0\,[x]_c^{c+2\pi} = 2\pi C_0$$

$$\therefore \qquad C_0 = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx \qquad \ldots (4)$$

Now taking the expanded form of (3), multiplying by e^{-inx} and integrating w.r.t x between $c,\ c+2\pi$ by using the basic integral evaluated earlier we have,

$$\int_c^{c+2\pi} f(x)\,e^{-inx}\,dx = C_n\int_c^{c+2\pi} 1\cdot dx = 2\pi C_n$$

$$\therefore \qquad C_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,e^{-inx}\,dx \qquad \ldots (5)$$

Similarly on multiplication of the expanded form of (3) by e^{inx} and integrating w.r.t x between $c,\ c+2\pi$ will give us

$$C_n' = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,e^{inx}\,dx \qquad \ldots (6)$$

Introducing the notation $C_n' = C_{-n}$ we have $C_n = \frac{a_n - ib_n}{2}$ for the range $n = 1$ to ∞ and $\frac{a_n + ib_n}{2}$ for the range $n = -\infty$ to -1; $C_n = C_0 = \frac{a_0}{2}$ also for $n = 0$.

With these, (3) can be represented in the form

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad \text{where} \quad C_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,e^{-inx}\,dx,$$

n is positive, negative or zero.

This is called the ***Complex form of the Fourier series*** or the ***exponential form of Fourier series*** and the C_n are called the ***Complex Fourier Coefficients.***

Also $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$ with $C_n = \frac{1}{2l}\int_c^{c+2l} f(x)\, e^{-in\pi x/l}\,dx$ is called as the complex form of the Fourier series of $f(x)$ having arbitrary period $2l$ where $c < x < c+2l$.

Note - 1 . *The working procedure for finding the complex form of Fourier series is same as discussed in the earlier articles. We need to evaluate* C_n *appropriately.*

Note - 2 . *We can as well get the usual form of Fourier series from the Complex form of Fourier series by taking the expression for* e^{inx} *or* $e^{in\pi x/l}$ *in the form* $e^{i\theta} = \cos\theta + i\sin\theta$

WORKED PROBLEMS

43. *Find the complex form of the Fourier series for* $f(x) = e^x$ *in* $-\pi < x < \pi$

>> The complex Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \quad \text{where}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cdot e^{-inx}\,dx$$

i.e.,
$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx = \frac{1}{2\pi(1-in)}\left[e^{(1-in)x}\right]_{-\pi}^{\pi}$$

$$= \frac{(1+in)}{2\pi(1+n^2)}\left\{e^{(1-in)\pi} - e^{-(1-in)\pi}\right\}$$

$$= \frac{(1+in)}{2\pi(1+n^2)}\left\{e^{\pi}(\cos n\pi - i\sin n\pi) - e^{-\pi}(\cos n\pi + i\sin n\pi)\right\}$$

$$= \frac{(1+in)}{2\pi(1+n^2)}\,(e^{\pi} - e^{-\pi})\cos n\pi = \frac{(1+in)\,2\sinh\pi\,(-1)^n}{2\pi(1+n^2)}$$

$$\mathbf{C_n = \frac{(1+in)(-1)^n \sinh\pi}{\pi(1+n^2)}}$$

Thus the required complex form of Fourier series is given by

$$f(x) = \frac{\sinh\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(1+in)(-1)^n}{1+n^2}\, e^{inx}$$

---

44. *Obtain the Fourier series in the complex form for* $f(x) = x$ *in* $-\pi < x < \pi$

>> $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$ where

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\left[x \cdot \frac{e^{-inx}}{-in} - 1 \cdot \frac{e^{-inx}}{i^2 n^2}\right]_{-\pi}^{\pi}, \text{ by Bernoulli's rule,}$$

$$= \frac{-1}{2\pi in}\left[x\, e^{-inx}\right]_{-\pi}^{\pi}, \text{ since the second term is zero.}$$

$$= \frac{-1}{2\pi in}\left(\pi e^{-in\pi} - (-\pi)\, e^{in\pi}\right) = \frac{-1}{2in}\left(e^{in\pi} + e^{-in\pi}\right) = \frac{i}{2n}\, 2\cos n\pi$$

$$C_n = \frac{i(-1)^n}{n}, \quad (n \neq 0)$$

Thus the required complex form of the Fourier series is given by

$$f(x) = i \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{(-1)^n}{n}\, e^{inx}$$

---

45. *Obtain the complex form of the Fourier series for the function*

$$f(x) = \begin{cases} -k & in \ -\pi < x < 0 \\ +k & in \ 0 < x < \pi \end{cases}$$

>> $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$, where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$

$$C_n = \frac{1}{2\pi}\left\{\int_{-\pi}^{0} -k\, e^{-inx}\, dx + \int_{0}^{\pi} k\, e^{-inx}\, dx\right\}$$

$$= \frac{k}{2\pi}\left\{\left[\frac{e^{-inx}}{in}\right]_{-\pi}^{0} + \left[\frac{e^{-inx}}{-in}\right]_{0}^{\pi}\right\}$$

$$= \frac{k}{2\pi in}\left\{(1 - e^{in\pi}) - (e^{-in\pi} - 1)\right\}$$

$$C_n = \frac{k}{2\pi i n}\left\{2-(e^{in\pi}+e^{-in\pi})\right\} = \frac{k}{2\pi i n}(2-2\cos n\pi)$$

$$C_n = \frac{k\left\{1-(-1)^n\right\}}{\pi i n}, \quad (n \neq 0)$$

Thus the required complex form of the Fourier series is given by

$$f(x) = \frac{k}{i\pi}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\left\{1-(-1)^n\right\}}{n}\, e^{inx}$$

---

46. *Obtain the complex form of the Fourier series for* $f(x) = e^{-x}$ *in* $(-1, 1)$

>> Here $2l = 2$ or $l = 1$. Hence we have,

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x} \text{ where } \quad C_n = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-in\pi x}\, dx$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x}\cdot e^{-in\pi x}\, dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx$$

$$C_n = \frac{1}{2}\cdot\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1}$$

$$= \frac{e^{-(1+in\pi)} - e^{(1+in\pi)}}{-2(1+in\pi)} = \frac{e^{(1+in\pi)} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$= \frac{e(\cos n\pi + i\sin n\pi) - 1/e(\cos n\pi - i\sin n\pi)}{2(1+in\pi)}$$

$$= \frac{(e-1/e)\cos n\pi}{2(1+in\pi)} = \frac{e^1 - e^{-1}}{2}\cdot\frac{(-1)^n}{1+in\pi}, \quad \text{since } \sin n\pi = 0$$

$$C_n = \frac{\sinh 1\,(-1)^n(1-in\pi)}{1+n^2\pi^2}$$

Thus the required complex form of Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\sinh 1\,(-1)^n(1-in\pi)}{1+n^2\pi^2}\, e^{in\pi x}$$

1.10 Practical Harmonic Analysis

So far, we have discussed the methods of obtaining the Fourier series of a known function $f(x)$ in a given interval. However, there will also be situations where in there will be no known functional expression for $f(x)$ but only the values at some equidistant points will be known.

Harmonic analysis is the process of finding the constant term and the first few cosine and sine terms numerically.

The Fourier series of period 2π of a function $y = f(x)$ will be of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx\, dx.$$

$a_0/2$ is called the constant term and the groups of terms $(a_1 \cos x + b_1 \sin x)$, $(a_2 \cos 2x + b_2 \sin 2x)$ etc. are called the *first harmonics, second harmonics* etc.

To derive the relevant formulae, we need the following principle.

The mean value of a continuous function $y = f(x)$ in the range (a, b) is given by
$$\frac{1}{b-a}\int_a^b f(x)\, dx$$

Suppose we have a set of N values of $y = f(x)$ having period 2π at equidistant points of x in the interval $c \le x < c + 2\pi$ or $c < x \le c + 2\pi$, [*if the values of y at $x = c$ and $x = c + 2\pi$ are given we must omit one of them. Infact $(y)_{x=c} = (y)_{x=c+2\pi}$ by the periodic property $f(x) = f(x + 2\pi)$*] the Fourier coefficients a_0, a_n, b_n, assume the following form by the earlier stated principle.

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\, dx = 2\left[\frac{1}{(c+2\pi)-c}\int_c^{c+2\pi} f(x)\, dx\right]$$

Here $a = c, b = c + 2\pi$

ie., $a_0 = 2$ [mean value of $f(x) = y$ in $(c, c + 2\pi)$]

ie., $a_0 = 2\left[\frac{\Sigma y}{N}\right]$ or $a_0 = \frac{2}{N}\Sigma y$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx = 2\left[\frac{1}{(c+2\pi)-c}\int_c^{c+2\pi} f(x)\cos nx\,dx\right]$$

$$a_n = 2[\text{mean value of } y\cos nx \text{ in } (c,\ c+2\pi)]$$

ie., $$a_n = 2\left[\frac{\sum y\cos nx}{N}\right] \quad \text{or} \quad a_n = \frac{2}{N}\sum y\cos nx$$

Similarly $$b_n = 2\left[\frac{\sum y\sin nx}{N}\right] \quad \text{or} \quad b_n = \frac{2}{N}\sum y\sin nx$$

Suppose we have a set of N values of $y = f(x)$ having period $2l$ at equidistant points of x in the interval $c \le x < c+2l$ or $c < x \le c+2l$

$$a_n = 2\left[\sum\frac{y\cos(n\pi x/l)}{N}\right];\ b_n = 2\left[\sum\frac{y\sin(n\pi x/l)}{N}\right].$$

Taking $\theta = \pi x/l$

$$a_n = \frac{2}{N}\sum y\cos n\theta,\ b_n = \frac{2}{N}\sum y\sin n\theta.$$

Note : *All these formulae holds good for half range Fourier series also.*

Working procedure for problems

- We have to first write down the period of $y = f(x)$ from the given range of the values of x.
- If the period is 2π, depending on the harmonics required we prepare the relevant table along with the summations $(\sum)$ of y; $y\cos x, y\cos 2x\ldots$; $, y\sin x, y\sin 2x\ldots$ and compute the harmonics using the formulae derived by taking $n = 1,\ 2,\ \ldots$.
- If the period is not 2π we equate it with $2l$ to obtain the value of l.
- The summations of y; $y\cos\theta$, $y\cos 2\theta,\ldots$; $y\sin\theta$, $y\sin 2\theta\ldots$where $\theta = \pi x/l$ will be required to compute the desired harmonics.

WORKED PROBLEMS

47. *Determine the constant term and the first cosine and sine terms of the Fourier series expansion of y from the following data.*

x^0	0	45	90	135	180	225	270	315
y	2	3/2	1	1/2	0	1/2	1	3/2

>> Here the interval of x is 0^0 to 360^0 That is $0 \le x < 2\pi$ .We have to find a_0 ; $(a_1,\ b_1)$. This requires the summation of y, $y\cos x$, $y\sin x$.

x^0	y	$\cos x$	$y \cos x$	$\sin x$	$y \sin x$
0	2.0	1	2.0	0	0
45	1.5	0.7071	1.06065	0.7071	1.06065
90	1.0	0	0	1	1.0
135	0.5	− 0.7071	− 0.35355	0.7071	0.35355
180	0	− 1	0	0	0
225	0.5	− 0.7071	− 0.35355	− 0.7071	− 0.35355
270	1.0	0	0	− 1	− 1
315	1.5	0.7071	1.06065	− 0.7071	− 1.06065
Totals	**8.0**		**3.4142**		**0**

$$a_0 = \frac{2}{N} \sum y, \quad a_1 = \frac{2}{N} \sum y \cos x, \quad b_1 = \frac{2}{N} \sum y \sin x, \quad N = 8.$$

Also from the table,

$$\sum y = 8.0, \quad \sum y \cos x = 3.4142, \quad \sum y \sin x = 0.$$

$$a_0 = \frac{2}{8}(8.0) = 2 \quad \text{or} \quad \frac{a_0}{2} = 1\,; \quad a_1 = \frac{2}{8}(3.4142) = 0.85355 \quad b_1 = \frac{2}{8}(0) = 0$$

The Fourier series of y up to the first harmonic is given by

$$y = a_0/2 + a_1 \cos x + b_1 \sin x$$

Thus $\mathbf{y = 1 + 0.85355 \cos x}$

---

48. *Obtain the Fourier series of y up to the second harmonics for the following values.*

x^0	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	− 1.5	0	2.8	3.4

>> The interval of x is $0 < x \leq 2\pi$ and period of $y = f(x)$ is 2π We have to compute $a_0\,;\ a_1,\ b_1\,; a_2,\ b_2$

$$a_0 = \frac{2}{N} \sum y, \quad a_1 = \frac{2}{N} \sum y \cos x, \quad b_1 = \frac{2}{N} \sum y \sin x$$

$$a_2 = \frac{2}{N} \sum y \cos 2x, \quad b_2 = \frac{2}{N} \sum y \sin 2x. \quad \text{Here } N = 8\,; \ \frac{2}{N} = \frac{1}{4}$$

x^0	y	$\cos x$	$y\cos x$	$\sin x$	$y\sin x$	$\cos 2x$	$y\cos 2x$	$\sin 2x$	$y\sin 2x$
45	4.0	0.7071	2.8284	0.7071	2.8284	0	0	1	4.0
90	3.8	0	0	1	3.8	−1	−3.8	0	0
135	2.4	−0.7071	−1.69704	0.7071	1.69704	0	0	−1	−2.4
180	2.0	−1	−2.0	0	0	1	2.0	0	0
225	−1.5	−0.7071	1.06065	−0.7071	1.06065	0	0	1	−1.5
270	0	0	0	−1	0	−1	0	0	0
315	2.8	0.7071	1.97988	−0.7071	−1.97988	0	0	−1	−2.8
360	3.4	1	3.4	0	0	1	3.4	0	0
Totals	**16.9**		**5.57189**		**7.40621**		**1.6**		**−2.7**

From the table,

$$\sum y = 16.9\,,\quad \sum y\cos x = 5.57189\,,\quad \sum y\sin x = 7.40621\,,$$

$$\sum y\cos 2x = 1.6\,,\quad \sum y\sin 2x = -2.7$$

$$a_0 = 1/4\cdot(16.9) = 4.225\,,\qquad a_0/2 = 2.1125,$$

$$a_1 = 1/4\cdot(5.57189) = 1.393\,.\qquad b_1 = 1/4\cdot(7.40621) = 1.8516$$

$$a_2 = 1/4\cdot(1.6) = 0.4\qquad b_2 = 1/4\cdot(-2.7) = -0.675$$

The Fourier series of y up to the second harmonic is given by

$$y = a_0/2 + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$$

Thus $\mathbf{y = 2.1125 + (1.393\cos x + 1.8516\sin x) + (0.4\cos 2x - 0.675\sin 2x)}$

---

49 . *Given the following table*

x^0	0	60^0	120^0	180^0	240^0	300^0
y	7.9	7.2	3.6	0.5	0.9	6.8

Obtain the Fourier series neglecting terms higher than first harmonics.

>> Here the interval of x is 0^0 to 360^0. That is $0 \le x < 2\pi$.

We are required to find a_0, a_1, b_1 only.

x^0	y	$\cos x$	$y \cos x$	$\sin x$	$y \sin x$
0	7.9	1	7.9	0	0
60	7.2	0.5	3.6	0.866	6.2352
120	3.6	− 0.5	− 1.8	0.866	3.1176
180	0.5	− 1	− 0.5	0	0
240	0.9	− 0.5	− 0.45	− 0.866	− 0.7794
300	6.8	0.5	3.4	− 0.866	− 5.8888
Totals	**26.9**	.	**12.15**	.	**2.6846**

Here $N = 6$; $2/N = 1/3$

$$a_0 = \frac{2}{N} \sum y = \frac{1}{3} (26.9) = 8.9667 \quad ; \quad \frac{a_0}{2} = 4.48335$$

$$a_1 = \frac{2}{N} \sum y \cos x = \frac{1}{3} (12.15) = 4.05$$

$$b_1 = \frac{2}{N} \sum y \sin x = \frac{1}{3} (2.6846) = 0.8949$$

The Fourier series upto the first harmonic is given by

$$y = a_0/2 + (a_1 \cos x + b_1 \sin x)$$

Thus $\mathbf{y = 4.48335 + (4.05 \cos x + 0.8949 \sin x)}$

---

50. *The turning moment T on the crank shaft of a steam engine for the crank angle θ is given as follows.*

θ^0	0	30	60	90	120	150	180	210	240	270	300	330
T	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2

Expand T as a Fourier series up to first harmonics.

>> Here the interval of θ is $0 \le \theta < 2\pi$. Period of T is 2π. We are required to find a_0, a_1, b_1. The corresponding formulae are

$$a_0 = \frac{2}{N} \sum T, \quad a_1 = \frac{2}{N} \sum T \cos\theta, \quad b_1 = \frac{2}{N} \sum T \sin\theta. \quad N = 12, \quad \frac{2}{N} = \frac{1}{6}$$

θ^0	T	$\cos\theta$	$T\cos\theta$	$\sin\theta$	$T\sin\theta$
0	0	1	0	0	0
30	2.7	0.866	2.3382	0.5	1.35
60	5.2	0.5	2.6	0.866	4.5032
90	7.0	0	0	1	7.0
120	8.1	– 0.5	– 4.05	0.866	7.0146
150	8.3	– 0.866	– 7.1878	0.5	4.15
180	7.9	– 1	– 7.9	0	0
210	6.8	– 0.866	– 5.8888	– 0.5	– 3.4
240	5.5	– 0.5	– 2.75	– 0.866	– 4.763
270	4.1	0	0	– 1	– 4.1
300	2.6	0.5	1.3	– 0.866	– 2.2516
330	1.2	0.866	1.0392	– 0.5	– 0.6
Totals	**59.4**		**– 20.4992**		**8.9032**

$$a_0 = \frac{1}{6}\sum T \quad = \frac{1}{6}(59.4) = 9.9 \quad ; \quad \frac{a_0}{2} = 4.95$$

$$a_1 = \frac{1}{6}\sum T\cos\theta = \frac{1}{6}(-20.4992) = -3.4165$$

$$b_1 = \frac{1}{6}\sum T\sin\theta = \frac{1}{6}(8.9032) = 1.4839$$

The Fourier series upto the first harmonic is given by

$$T = f(\theta) = a_0/2 + (a_1\cos\theta + b_1\sin\theta)$$

Thus $\mathbf{T = 4.95 - 3.4165\cos\theta + 1.4839\sin\theta}$

---

51. *Express y as a Fourier series upto the third harmonics given*

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.98	1.30	1.05	1.30	– 0.88	– 0.25	1.98

>> Here the interval of x is $(0 \le x \le 2\pi)$ and the values of y at $x = 0$ and $x = 2\pi$ must be same by the periodic property $f(x+2\pi) = f(x)$. In the given problem the values of y at $x = 0$ and 2π both are given and we must omit one of them. Let us omit the last value. The values of x in degrees are 0, 60, 120, 180, 240, 300 and $N = 6$.

The relevant table is formulated by considering the values of $\cos x$, $\sin x$; $\cos 2x$, $\sin 2x$; $\cos 3x$, $\sin 3x$ rounded off to three places of decimals and then multiplied by the values of y.

0	y	$y\cos x$	$y\cos 2x$	$y\cos 3x$	$y\sin x$	$y\sin 2x$	$y\sin 3x$
0	1.98	1.98	1.98	1.98	0	0	0
60	1.3	0.65	− 0.65	− 1.3	1.1258	1.1258	0
120	1.05	− 0.525	− 0.525	1.05	0.9093	− 0.9093	0
180	1.3	− 1.3	1.3	− 1.3	0	0	0
240	− 0.88	0.44	0.44	− 0.88	0.76208	− 0.76208	0
300	− 0.25	− 0.125	0.125	0.25	0.2165	0.2165	0
Totals	**4.5**	**1.12**	**2.67**	**− 0.2**	**3.01368**	**− 0.32908**	**0**

$$a_0 = \frac{2}{N}\sum y = \frac{1}{3}(4.5) = 1.5 \quad ; \quad \frac{a_0}{2} = 0.75$$

$$a_1 = \frac{2}{N}\sum y\cos x = \frac{1}{3}(1.12) = 0.3733$$

$$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{1}{3}(2.67) = 0.89$$

$$a_3 = \frac{2}{N}\sum y\cos 3x = \frac{1}{3}(-0.2) = -0.0667$$

$$b_1 = \frac{2}{N}\sum y\sin x = \frac{1}{3}(3.01368) = 1.00456$$

$$b_2 = \frac{2}{N}\sum y\sin 2x = \frac{1}{3}(-0.32908) = -0.1097$$

$$b_3 = \frac{2}{N}\sum y\sin 3x = \frac{1}{3}(0) = 0.$$

Fourier series up to third harmonics is given by

$$y = a_0/2 + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + (a_3\cos 3x + b_3\sin 3x)$$

Thus $\mathbf{y = 0.75 + (0.3733\ \cos x + 1.00456\ \sin x)}$

$$\mathbf{+ (0.89\cos 2x - 0.1097\sin 2x) + (-0.0667\cos 3x)}$$

52. *Find the Fourier series to represent y (x) upto the second harmonic from the follwing data :*

x^0	30	60	90	120	150	180	210	240	270	300	330	360
y	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

>> The period of $y(x)$ is $2\pi = 360^0$ and the Fourier series upto the second harmonic is given by

$$y(x) = a_0/2 + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

where $a_0 = \frac{2}{N} \sum y, \quad a_1 = \frac{2}{N} \sum y \cos x, \quad a_2 = \frac{2}{N} \sum y \cos 2x$

$$b_1 = \frac{2}{N} \sum y \sin x, \quad b_2 = \frac{2}{N} \sum y \sin 2x \ ;$$

Here $N = 12$ and we prepare the following table.

x^0	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
30	2.34	0.87	0.5	0.5	0.87	2.0358	1.17	1.17	2.0358
60	3.01	0.5	− 0.5	0.87	0.87	1.505	− 1.505	2.6187	2.6187
90	3.68	0	− 1	1	0	0	− 3.68	3.68	0
120	4.15	− 0.5	− 0.5	0.87	− 0.87	− 2.075	− 2.075	3.6105	− 3.6105
150	3.69	− 0.87	0.5	0.5	− 0.87	− 3.2103	1.845	1.845	− 3.2103
180	2.20	− 1	1	0	0	− 2.2	2.2	0	0
210	0.83	− 0.87	0.5	− 0.5	0.87	− 0.7221	0.415	− 0.415	0.7221
240	0.51	− 0.5	− 0.5	− 0.87	0.87	− 0.255	-0.255	-0.4437	0.4437
270	0.88	0	− 1	− 1	0	0	− 0.88	− 0.88	0
300	1.09	0.5	− 0.5	− 0.87	− 0.87	0.545	− 0.545	− 0.9483	− 0.9483
330	1.19	0.87	0.5	− 0.5	− 0.87	1.0353	0.595	− 0.595	− 1.0353
360	1.64	1	1	0	0	1.64	1.64	0	0
Totals	**25.21**					**− 1.7013**	**− 1.075**	**9.6422**	**− 2.9841**

$$a_0 = \frac{25.21}{6} = 4.2017 \ ; \ \frac{a_0}{2} = 2.1$$

$$a_1 = \frac{-1.7013}{6} = -0.28, \quad a_2 = \frac{-1.075}{6} = -0.18$$

$$b_1 = \frac{9.6422}{6} = 1.61, \quad b_2 = \frac{-2.9841}{6} = -0.5$$

Thus the required Fourier series upto the second harmonic is given by

$$\mathbf{y(x) = 2.1 + (-0.28\cos x + 1.61\sin x) + (-0.18\cos 2x - 0.5\sin 2x)}$$

---

53. *Compute the first two harmonics of the Fourier series of $f(x)$ given the following table*

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

>> We have values of $f(x) = y$ in the interval $0 \le x \le 2\pi$ and hence we omit the last value $f(2\pi) = 1.0$ which is same as $f(0)$. The relevant table for computing the first two harmonics is as follows.

x^0	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1	1	1	0	0	1	1	0	0
60	1.4	0.5	− 0.5	0.866	0.866	0.7	− 0.7	1.2124	1.2124
120	1.9	− 0.5	− 0.5	0.866	− 0.866	− 0.95	− 0.95	1.6454	− 1.6454
180	1.7	− 1	1	0	0	− 1.7	1.7	0	0
240	1.5	− 0.5	− 0.5	− 0.866	0.866	− 0.75	− 0.75	− 1.299	1.299
300	1.2	0.5	− 0.5	− 0.866	− 0.866	0.6	− 0.6	− 1.0392	− 1.0392
Totals						**1.1**	**− 0.3**	**0.5196**	**− 0.1732**

$$a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{6}(-1.1) = -0.367$$

$$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{2}{6}(-0.3) = -0.1$$

$$b_1 = \frac{2}{N}\sum y\sin x = \frac{2}{6}(0.5196) = 0.1732$$

$$b_2 = \frac{2}{N}\sum y\sin 2x = \frac{2}{6}(-0.1732) = -0.0577$$

The first two harmonics are

$(a_1\cos x + b_1\sin x)$ and $(a_2\cos 2x + b_2\sin 2x)$. Thus they are

$$\mathbf{(-0.367\cos x + 0.1732\sin x) \text{ and } (-0.1\cos 2x - 0.0577\sin 2x)}$$

---

54. *Obtain the constant term and the coefficients of the first cosine and sine terms in the Fourier expansion of y from the table*

x	0	1	2	3	4	5
y	9	18	24	28	26	20

>> The values at 0, 1, 2, 3, 4, 5 are given ($N = 6$) and hence the interval of x should be $0 \le x < 6$.

$\therefore$ length of the interval is $6 - 0 = 6$. Comparing with $2l$ we have $2l = 6$. or $l = 3$.

The Fourier series of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Since $l = 3$, the series containing the first harmonics is

$$y = f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}$$

Writing $\frac{\pi x}{3} = \theta$, $y = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$; $N = 6$ and $\frac{2}{N} = \frac{1}{3}$

x	$\theta = \pi x/3$	y	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	0	9	1	9	0	0
1	60^0	18	0.5	9	0.866	15.588
2	120^0	24	– 0.5	– 12	0.866	20.784
3	180^0	28	– 1	– 28	0	0
4	240^0	26	– 0.5	– 13	– 0.866	– 22.516
5	300^0	20	0.5	10	– 0.866	– 17.32
Total		**125**		**– 25**		**– 3.464**

$$a_0 = \frac{2}{N}\sum y = \frac{1}{3}(125) \approx 41.67 \; ; \quad \frac{a_0}{2} = 20.835$$

$$a_1 = \frac{2}{N}\sum y\cos\theta = \frac{1}{3}(-25) \approx -8.333$$

$$b_1 = \frac{2}{N}\sum y\sin\theta = \frac{1}{3}(-3.464) \approx -1.155$$

Constant term $= a_0/2 =$ **20.835**

Coefficient of the first cosine term $= a_1 =$ **– 8.333**

Coefficient of the first sine term $= b_1 =$ **– 1.155**

4

55. *Express y as a Fourier series up to the third harmonics given the following values.*

x	0	1	2	3	4	5
y	4	8	15	7	6	2

>> As in the previous problem the interval of x is $0 \le x < 6$

$\therefore \quad 2l = 6$ or $l = 3, N = 6 \;;\; 2/N = 1/3$

Fourier series up to the third harmonics is given by

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right)$$

$$+ \left(a_3 \cos\frac{3\pi x}{l} + b_3 \sin\frac{3\pi x}{l}\right), \quad \text{where } l = 3$$

$$\therefore \quad y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right)$$

$$+ \left(a_3 \cos\frac{3\pi x}{3} + b_3 \sin\frac{3\pi x}{3}\right)$$

Putting $\pi x/3 = \theta$

$$y = a_0/2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta)$$

x	$\theta = \pi x/3$	y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$	$y\sin\theta$	$y\sin 2\theta$	$y\sin 3\theta$
0	0	4	4	4	4	0	0	0
1	60^0	8	4	− 4	− 8	6.928	6.928	0
2	120^0	15	− 7.5	− 7.5	15	12.99	− 12.99	0
3	180^0	7	− 7	7	− 7	0	0	0
4	240^0	6	− 3	− 3	6	− 5.196	5.196	0
5	300^0	2	1	− 1	− 2	− 1.732	− 1.732	0
Total		**42**	**− 8.5**	**− 4.5**	**8**	**12.99**	**− 2.598**	0

$$a_0 = \frac{2}{N}\sum y = \frac{1}{3}(42) = 14 \;;\; \frac{a_0}{2} = 7$$

$a_1 = \frac{2}{N} \Sigma\, y \cos\theta = \frac{1}{3}(-8.5) = -2.833 \qquad b_1 = \frac{2}{N} \Sigma\, y \sin\theta = \frac{1}{3}(12.99) = 4.33$

$a_2 = \frac{2}{N} \Sigma\, y \cos 2\theta = \frac{1}{3}(-4.5) = -1.5 \qquad b_2 = \frac{2}{N} \Sigma\, y \sin 2\theta = \frac{1}{3}(-2.598) = -0.866$

$a_3 = \frac{2}{N} \Sigma\, y \cos 3\theta = \frac{1}{3}(8) \approx 2.667 \qquad b_3 = \frac{2}{n} \Sigma\, y \sin 3\theta = \frac{1}{3}(0) = 0$

The required Fourier series upto the third harmonics is given by

$$\mathbf{y = 7 - 2.833 \cos \frac{\pi x}{3} + 4.33 \sin \frac{\pi x}{3} - 1.5 \cos \frac{2\pi x}{3} - 0.866 \sin \frac{2\pi x}{3} + 2.667 \cos \pi x}$$

---

56. *Obtain the constant term and the first three coefficients in the Fourier cosine series for y using the following table*

x	0	1	2	3	4	5
y	4	8	15	7	6	2

>> Here the interval of x is $0 \le x < 6$ and since the coefficients of the Fourier cosine series are to be found we have to conclude that it should be the cosine half range Fourier series of $y = f(x)$ in $(0, 6)$. Comparing with half the range $(0, l)$ we get $l = 6$ and the Fourier cosine series is of the form

$$y = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos \frac{n\pi x}{l}.$$

ie., $$y = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{6}\right) + a_2 \cos\left(\frac{2\pi x}{6}\right) + a_3 \cos\left(\frac{3\pi x}{6}\right) + \ldots$$

Putting $\pi x/6 = \theta$ the Fourier series upto third harmonics assumes the form

$$y = a_0/2 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta \qquad \ldots (1)$$

We have to compute $a_0/2$ *(cosntant term)*, a_1, a_2, a_3, by the formulae

$$a_0 = \frac{2}{N} \Sigma\, y, \quad a_1 = \frac{2}{N} \Sigma\, y \cos\theta, \quad a_2 = \frac{2}{N} \Sigma\, y \cos 2\theta, \quad a_3 = \frac{2}{N} \Sigma\, y \cos 3\theta$$

Here $N = 6$; $\frac{2}{N} = \frac{1}{3}$

The relevant table is as follows :

x	θ	y	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0	0	4	1	1	1	4	4	4
1	30^0	8	0.866	0.5	0	6.928	4	0
2	60^0	15	0.5	-0.5	-1	7.5	-7.5	-15
3	90^0	7	0	-1	0	0	-7	0
4	120^0	6	-0.5	-0.5	1	-3	-3	6
5	150^0	2	-0.866	0.5	0	-1.732	1	0
Sum		**42**				**13.696**	**-8.5**	**-5**

Here, $\sum y = 42$, $\sum y\cos\theta = 13.696$, $\sum y\cos 2\theta = -8.5$, $\sum y\cos 3\theta = -5$

$$a_0 = \frac{1}{3}(42) = 14 \quad \therefore \quad \frac{a_0}{2} = 7; \quad a_1 = \frac{1}{3}(13.696) \approx 4.565$$

$$a_2 = \frac{1}{3}(-8.5) \approx -2.833; \quad a_3 = \frac{1}{3}(-5) \approx -1.667$$

The required values $a_0/2$, a_1, a_2, a_3 are respectively

7, 4.565, -2.833 and -1.667

---

57. *Obtain the constant term and the coefficients of* $\sin\theta$ *and* $\sin 2\theta$ *in the Fourier expansion of y given the following data*

θ^0	0	60	120	180	240	300	360
y	0	9.2	14.4	17.8	17.3	11.7	0

>> Here the interval of θ is $(0, 360^0)$. That is $0 \le \theta \le 2\pi$ and the value of y at $\theta = 0$ and $\theta = 2\pi$ must be the same by the periodic property $f(\theta + 2\pi) = f(\theta)$. When the values are given both at $\theta = 0$ and $\theta = 2\pi$ we must omit one of them. We need to compute the coefficient of : $\sin\theta$ being b_1 and $\sin 2\theta$ being b_2 in the Fourier expansion of y using the formulae,

$b_1 = \frac{2}{N}\sum y\sin\theta$, $b_2 = \frac{2}{N}\sum y\sin 2\theta$, where $N = 6$ by omitting the last value. The relevant table is as follows.

θ^0	y	$\sin\theta$	$\sin 2\theta$	$y\sin\theta$	$y\sin 2\theta$
0	0	0	0	0	0
60	9.2	0.866	0.866	7.9672	7.9672
120	14.4	0.866	– 0.866	12.4704	– 12.4704
180	17.8	0	0	0	0
240	17.3	– 0.866	0.866	– 14.9818	14.9818
300	11.7	– 0.866	– 0.866	– 10.1322	– 10.1322
Total				**– 4.6764**	**0.3464**

Thus we have

$$b_1 = \frac{2}{6}(-4.6764) = \mathbf{-1.5588} \qquad b_2 = \frac{2}{6}(0.3464) = \mathbf{0.1155}$$

---

58. *The following values of y and x are given. Find the Fourier series of y upto second harmonics.*

x	0	2	4	6	8	10	12
y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

>> The values of y at $x = 0$ and $x = 12$ are same. Hence the interval of x is $(0, 12)$. That is $0 \le x \le 12$ and we shall omit the value of y for $x = 12$ in the process of calculation.

The Fourier series of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Putting $l = 6$, the Fourier series up to the second harmonics is given by

$$y = f(x) = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{6} + b_1\sin\frac{\pi x}{6}\right) + \left(a_2\cos\frac{2\pi x}{6} + b_2\sin\frac{2\pi x}{6}\right)$$

Putting $\theta = \pi x/6$ we have

$$y = a_0/2 + (a_1\cos\theta + b_1\sin\theta) + (a_2\cos 2\theta + b_2\sin 2\theta).$$

The relevant table is as follows :

x	y	$\theta^0 = \pi x/6$	$\cos\theta$	$y\cos\theta$	$\cos 2\theta$	$y\cos 2\theta$	$\sin\theta$	$y\sin\theta$	$\sin 2\theta$	$y\sin 2\theta$
0	9.0	0	1	9	1	9.0	0	0	0	0
2	18.2	60	0.5	9.1	– 0.5	– 9.1	0.866	15.7612	0.866	15.7612
4	24.4	120	– 0.5	– 12.2	– 0.5	– 12.2	0.866	21.1304	– 0.866	– 21.1304
6	27.8	180	– 1	– 27.8	1	27.8	0	0	0	0
8	27.5	240	– 0.5	– 13.75	– 0.5	– 13.75	– 0. 866	– 23.815	0.866	23.815
10	22.0	300	0.5	11.0	– 0.5	– 11.0	– 0.866	– 19.052	– 0.866	– 19.052
Total	**128.9**			**– 24.65**		**– 9.25**	.	**– 5.9754**		**– 0.6062**

$$a_0 = \frac{2}{N}\sum y \qquad = \frac{2}{6}(128.9) \approx 42.967 \quad \therefore \quad \frac{a_0}{2} \approx 21.4835$$

$$a_1 = \frac{2}{N}\sum y\cos\theta \qquad = \frac{2}{6}(-24.65) \approx -8.217$$

$$a_2 = \frac{2}{N}\sum y\cos 2\theta \qquad = \frac{2}{6}(-9.25) \approx -3.083$$

$$b_1 = \frac{2}{N}\sum y\sin\theta \qquad = \frac{2}{6}(-5.9754) \approx -1.9918$$

$$b_2 = \frac{2}{N}\sum y\sin 2\theta \qquad = \frac{2}{6}(-0.6062) \approx -0.202$$

The required Fourier series up to the second harmonics is given by

$$y = f(x) = \mathbf{21.4835} + \left(-\mathbf{8.217}\cos\frac{\pi x}{6} - \mathbf{1.9918}\sin\frac{\pi x}{6}\right) + \left(-\mathbf{3.083}\cos\frac{\pi x}{3} - \mathbf{0.202}\sin\frac{\pi x}{3}\right)$$

---

59. *The following data gives the variations of a periodic current over a period.*

t *secs*	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A *amps*	9.0	18.2	24.4	27.8	27.5	22.0	9.0

Find numerically the direct current part of the variable current and obtain the amplitudes upto the second harmonic.

>> We observe that the values of A at $t = 0$ and $t = T$ are the same. Hence we shall omit the last value. We convert $A = f(t)$ to the period 2π by putting $\theta = 2\pi(t/T)$ so that we have $\theta = 0$ when $t = 0$ and $\theta = 2\pi$ when $t = T$. The corresponding values of θ are respectively $0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ, 360^\circ$

The Fourier series upto the second harmonics is represented by

$$A = a_0/2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta)$$

We prepare the relevant table considering the values of A and θ in $0 \le \theta < 2\pi$.

Since the data is same as in Problem - 58 the Fourier series is given by

$$A = f(\theta) = 21.4835 + (-8.217 \cos\theta - 1.9918 \sin\theta)$$

$$+ (-3.083 \cos 2\theta - 0.202 \sin 2\theta)$$

The *direct current part* of the variable current is the constant term in the Fourier series being **21.4835**

Amplitude of the first harmonic $= \sqrt{a_1^2 + b_1^2} = \mathbf{8.455}$

Amplitude of the second harmonic $= \sqrt{a_2^2 + b_2^2} = \mathbf{3.09}$

1.11

ADDITIONAL PROBLEMS

[From the previous VTU question papers]

60. *Expand* $f(x) = e^{-x}$ *as a Fourier series in the interval* $(-l, l)$

>> The Fourier series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \; ; -l < x < l$$

Here we have,

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\, dx = \frac{1}{l}\left[-e^{-x}\right]_{-l}^{l} = \frac{-1}{l}\,(e^{-l} - e^{l}) = \frac{e^{l} - e^{-l}}{l} = \frac{2\sin hl}{l}$$

$$a_0/2 = 2\sin hl/l$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\,dx \quad \text{and by a standard formula,}$$

$$a_n = \frac{1}{l}\left[\frac{e^{-x}}{1+(n\pi/l)^2}\left\{-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right\}\right]_{-l}^{l}$$

$$= \frac{-l}{l^2+n^2\pi^2}\left[e^{-l}\cos n\pi - e^{l}\cos n\pi\right] \quad \text{since } \sin n\pi = 0\,.$$

$$a_n = \frac{l\cos n\pi\,(e^{l}-e^{-l})}{l^2+n^2\pi^2} = \frac{2l\,(-1)^n\sin hl}{l^2+n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx \quad \text{and by a standard formula}$$

$$b_n = \frac{1}{l}\left[\frac{e^{-x}}{1+(n\pi/l)^2}\left\{-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right\}\right]_{-l}^{l}$$

$$= \frac{-l}{l^2+n^2\pi^2}\cdot\frac{n\pi}{l}\,(e^{-l}\cos n\pi - e^{l}\cos n\pi)$$

$$b_n = \frac{n\pi\cos n\pi\,(e^{l}-e^{-l})}{l^2+n^2\pi^2} = \frac{2n\pi\,(-1)^n\sin hl}{l^2+n^2\pi^2}$$

Thus the required Fourier series is given by

$$e^{-x} = \frac{\sin hl}{l} + \sum_{n=1}^{\infty}\frac{2l\,(-1)^n\sin hl}{l^2+n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{2n\pi\,(-1)^n\sin hl}{l^2+n^2\pi^2}\sin\frac{n\pi x}{l}$$

- -

61. *Find the half range cosine series for the function*

$$f(x) = \begin{cases} kx & in \quad 0 \le x \le l/2 \\ k(l-x) & in \quad l/2 \le x \le l \end{cases} \quad \textit{where k is a constant.}$$

>> The cosine half range Fourier series of $f(x)$ in $0 \le x \le l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

Now $\quad a_0 = \frac{2k}{l}\left[\int_0^{l/2} x\,dx + \int_{l/2}^{l}(l-x)\,dx\right]$

$$= \frac{2k}{l}\left\{\left[\frac{x^2}{2}\right]_0^{l/2} + \left[lx - \frac{x^2}{2}\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\left\{\left(\frac{l^2}{8} - 0\right) + \left(l^2 - \frac{l^2}{2}\right) - \left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right\} = \frac{2k}{l}\cdot\frac{l^2}{4} = \frac{kl}{2}$$

$\mathbf{a_0 / 2 = kl / 4}$

$$a_n = \frac{2k}{l}\left\{\int_0^{l/2} x\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\cos\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2k}{l}\left\{\left[x\cdot\frac{\sin\frac{n\pi x}{l}}{n\pi/l} - 1\cdot\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)^2}\right]_0^{l/2}\right.$$

$$\left. + \left[(l-x)\frac{\sin\frac{n\pi x}{l}}{n\pi/l} - (-1)\cdot\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)^2}\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\left\{\frac{l}{n\pi}\left(\frac{l}{2}\sin\frac{n\pi}{2} - 0\right) + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right.$$

$$\left. + \frac{l}{n\pi}\left(0 - \frac{l}{2}\sin\frac{n\pi}{2}\right) - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right\}$$

$$= \frac{2k}{l}\cdot\frac{l^2}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right)$$

$$\mathbf{b_n = \frac{2kl}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - [1 + (-1)^n]\right\}}$$

Thus the required cosine half range Fourier series is given by

$$\mathbf{f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{2\cos\frac{n\pi}{2} - [1 + (-1)^n]\right\}\cos\frac{n\pi x}{l}}$$

---

62. *The following table gives the variations of a periodic current A over a period T*

t (*sec*)	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A (*amp*)	1.98	1.30	1.05	1.30	− 0.88	− 0.25	1.98

Show that there is a constant part of 0.75 amp in the current A and also obtain the amplitude of the first harmonic.

>> We convert $A = f(t)$ to period 2π by putting $\theta = 2\pi(t/T)$. The corresponding values of θ are respectively $0°$, $60°$, $120°$, $180°$, $240°$, $300°$ and $360°$.

Since the value of A at $\theta = 0°$ and $\theta = 360°$ are the same, we omit the last value.

We need to prepare the table with column heads $\theta°$, A ; $\cos\theta$, $\sin\theta$; $A\cos\theta$, $A\sin\theta$ and equip with ΣA, $\Sigma A\cos\theta$ and $\Sigma A\sin\theta$.

[*Refer Problem*-51 *for computations*]

We have obtained $a_0 = 0.75$; $a_1 = 0.3733$, $b_1 = 1.00456$.

This shows that the **constant part in the current A is 0.75 amp.**, being the constant term a_0 of the Fourier series.

Also the amplitude of the first harmonic $= \sqrt{a_1^2 + b_1^2}$

Amplitude is given by $\sqrt{(0.3733)^2 + (1.00456)^2} = \mathbf{1.07168}$

---

63. *Express y in a Fourier series upto second harmonics given*

x^0	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	0.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

>> The period of $y = f(x)$ is $2\pi = 360^0$ and we prepare the following table.

Here $N = 12$

x^0	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	1.8	1	1	0	0	1.8	1.8	0	0
30	1.1	0.87	0.5	0.5	0.87	0.957	0.55	0.55	0.957
60	0.3	0.5	− 0.5	0.87	0.87	0.15	− 0.15	0.261	0.261
90	0.16	0	− 1	1	0	0	− 0.16	0.16	0
120	0.5	− 0.5	− 0.5	0.87	− 0.87	− 0.25	− 0.25	0.435	− 0.435
150	1.3	− 0.87	0.5	0.5	− 0.87	− 1.131	0.65	0.65	− 1.131
180	2.16	− 1	1	0	0	− 2.16	2.16	0	0
210	1.25	− 0.87	0.5	− 0.5	0.87	− 1.0875	0.625	− 0.625	1.0875
240	1.3	− 0.5	− 0.5	− 0.87	0.87	− 0.65	− 0.65	− 1.131	1.131
270	1.52	0	− 1	− 1	0	0	− 1.52	− 1.52	0
300	1.76	0.5	− 0.5	− 0.87	− 0.87	0.88	− 0.88	− 1.5312	− 1.5312
330	2	0.87	0.5	− 0.5	− 0.87	1.74	1	− 1	− 1.74
Total	**15.15**					**0.2485**	**3.175**	**− 3.7512**	**− 1.4007**

$$a_0 = \frac{2}{N} \sum y = \frac{2}{12} (15.15) = 2.525 \; ; \; a_0/2 = 1.2625$$

$$a_1 = \frac{2}{N} \sum y \cos x = \frac{2}{12} (0.2485) = 0.0414$$

$$a_2 = \frac{2}{N} \sum y \cos 2x = \frac{2}{12} (3.175) = 0.5292$$

$$b_1 = \frac{2}{N} \sum y \sin x = \frac{2}{12} (-3.7512) = -0.6252$$

$$b_2 = \frac{2}{N} \sum y \sin 2x = \frac{2}{12} (-1.4007) = -0.2335$$

The Fourier series of $y = f(x)$ upto second harmonics is given by

$$y = a_0/2 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$

Thus, $\mathbf{y = 1.2625 + (0.0414 \cos x - 0.6252 \sin x) + (0.5292 \cos 2x - 0.2335 \sin 2x)}$

EXERCISES

1. Analyse the following data harmonically upto the second harmonic.

x^0	0	30	60	90	120	150	180
y	134	444	326	106	94	144	66
x^0	210	240	270	300	330		
y	– 44	– 126	– 306	– 494	– 344		

2. Determine the first two harmonics in the Fourier series of y given the data

x^0	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	0.8	0.6	0.4	0.7	0.9	1.1	0.8

3. Express y as a Fourier series upto the second harmonic given

x^0	0	60	120	180	240	300	360
y	4	3	2	4	5	6	4

4. In a machine the displacement y of a given point is given for certain angles of θ.

θ^0	0	30	60	90	120	150	180
y	7.9	8	7.2	5.6	3.6	1.7	0.5
θ^0	210	240	270	300	330		
y	0.2	0.9	2.5	4.7	6.8		

Find the coefficients of $\sin \theta$ and $\sin 2\theta$ in the Fourier expansion of y.

5. Determine the first harmonic of the Fourier series for the values

x^0	0	30	60	90	120	150	180
y	– 3	2.07	5.29	9.57	10.2	4.68	3
x^0	210	240	270	300	330	360	
y	6.12	6.3	2.25	– 3.79	– 6.87	– 3	

6. The values of y, a periodic function of x, are given below for twelve equidistant values of x covering the whole period. Express y in a Fourier series as far as the third harmonics if the first value is for $x = 30^0$.
1.8, 1.1, 0.3, 0.16, 0.5, 1.5, 2.16, 1.88, 1.25, 1.3, 1.76, 2.
(*Values of the Fourier coefficients be given correct to two decimals*)

7. The turning moment T on the crank shaft of a steam engine for the crank angle θ is given

θ^0	0	15	30	45	60	75	90
T	0	2.7	5.2	7	8.1	8.3	7.9
θ^0	105	120	135	150	165		
T	6.8	5.5	4.1	2.6	1.2		

Exapand T as a series of sine upto the second harmonics.

8. Obtain the constant term and the coefficients of $\cos\theta$ and $\cos 2\theta$ in the Fourier expansion of y given the data

θ^0	0	45	90	135	180	225	270	315	360
y	5	4	2	1	−5	−4	−2	−1	5

9. Analyse the following data upto the first harmonic.

x	0	1	2	3	4	5	6	7	8	9	10	11
y	7	9	11	13	14	8	−7	−9	−11	−13	−14	−8

10. Obtain the constant term and the first three coefficients in the Fourier cosine series of y using the following table.

x	0	1	2	3	4	5
y	8	6	4	7	9	11

ANSWERS

1. $(300\sin x) + (100\cos 2x + 173\sin 2x)$

2. $(0.1\cos x - 0.289\sin x)$ and $(0\cdot\cos 2x + 0\cdot\sin 2x)$

3. $4 + (0.33\cos x - 1.732\sin x)$ [second harmonic is zero]

4. $b_1 = 1.492;\ b_2 = -0.072$

5. $-4.502 \cos x + 3.1718 \sin x$

6. $1.31 + (-0.07 \cos x - 0.62 \sin x) + (0.64 \cos 2x - 0.18 \sin 2x)$
$+ (-0.11 \cos 3x - 0.02 \sin 3x)$

7. $7.837 \sin \theta + 1.484 \sin 2\theta$

8. Const. term = 0, Coeff. of $\cos \theta = 3.56$, Coeff. of $\cos 2\theta = 0$

9. $0 + \left[2.122 \cos (\pi x/6) + 14.384 \sin (\pi x/6) \right]$

10. 7.5, 0.39, 1, 4.333